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Reconstructing a pure state of a spin s through three Stern-Gerlach measurements

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Consider a spin s prepared in a *pure* state. It is shown that, generically, the moduli of the $(2s + 1)$ spin components along three directions in space determine the state unambiguously. These probabilities are accessible experimentally by means of a standard Stern-Gerlach apparatus. To reconstruct a pure state is therefore possible on the basis of $3(2s + 1)$ measured intensities.

The reconstruction of a particle density-operator is possible in principle through repeated measurements on an ensemble of identically prepared systems [1,2]. Quantum states of vibrating molecules [3], of trapped ions [4], as well as the state of atoms in motion [5] have been reconstructed successfully in the laboratory. Similarly, quantum optical experiments [6] have been performed.

For a spin of length s , this question arises for states in a Hilbert space of finite dimension. There is an explicit expression for the density matrix ρ in terms of the moduli of spin components along $(4s + 1)$ appropriate directions in space [7]. This number can be reduced to $(2s + 1)$ upon adopting a different approach [8]. A standard Stern-Gerlach apparatus with variable orientation in space provides the corresponding probabilities in an experiment. Alternatively, a Wigner function defined on the discrete phase space associated with a finite-dimensional Hilbert space allows one to reconstruct quantum states [9]. This method has been adapted in [10] in order to determine a quantized electromagnetic mode of a cavity. Every proposed method of state reconstruction is bound to reflect on the link between the outcomes of a finite number of measurements obtained in an actual experiment and the mathematical probabilities which refer to *infinite* ensembles (see [11], for example).

Suppose now that the spin state to be reconstructed is known to be prepared in a *pure* state which is determined by less parameters than a mixed one. How to exploit this additional knowledge in the most efficient way? Reconstruction of pure states has been turned into a question as early as 1933 for a *particle* by Pauli [12] who did not provide an answer. One solution of the spin version of the problem [13] makes use of a *Feynman filter*. This is an advanced version of a Stern-Gerlach apparatus which is assumed to reveal the relative phases of the expansion coefficients of a pure spin state. Another approach relates expectation values of spin multipoles with the parameters which define the quantum state [14].

As shown in this letter, the pure state of a spin s is determined unambiguously if the *intensities* of the spin components are measured along *three* axes. Compared to the $(2s + 1)$ axes required for a mixed state [8], the experimental effort to perform state reconstruction is thus reduced considerably for large spins. Further, this result

is satisfactory from a mathematical point of view since it generalizes an earlier result: the intensities along two *infinitesimally close* axes spanning a plane define a unique pure state when complemented by the expectation value of a spin component “out of plane” [15]. Effectively, this means to measure $(2s + 1)$ probabilities along a third direction.

The states of a spin of magnitude s live in a Hilbert space \mathbf{H}^s of complex dimension $(2s + 1)$, which carries an irreducible representation of the group $SU(2)$. The components of the spin operator $\tilde{\mathbf{S}} \equiv \hbar \mathbf{s}$ with standard commutation relations $[\mathbf{s}_x, \mathbf{s}_y] = i\mathbf{s}_z, \dots$ generate rotations about the corresponding axes. The standard basis of the space \mathbf{H}^s is given by the eigenvectors of the z component of the spin, denoted by $|s, \mu_z\rangle$, $-s \leq \mu_z \leq s$. The transformation under the anti-unitary time reversal operator T fixes their phases, $T|s, \mu_z\rangle = (-1)^{s-\mu_z}|s, -\mu_z\rangle$. When expanded in the z basis ($\mu_k \equiv \mu_z$),

$$|\psi\rangle = \sum_{\mu_k=-s}^s \psi_{\mu_k} |s, \mu_k\rangle, \quad k = x, y, z, \quad (1)$$

a pure state is seen to be determined by $(2s + 1)$ complex coefficients $\psi_{\mu_z} \equiv \langle s, \mu_z | \psi \rangle$. If normalized, rays $|\psi\rangle$ depend on $4s$ real parameters. Two other bases of the space \mathbf{H}^s are used in Eq. (1): the sets $\{|s, \mu_x\rangle\}$ and $\{|s, \mu_y\rangle\}$ with $-s \leq \mu_x, \mu_y \leq s$, made up from the eigenvectors of the spin components \mathbf{s}_x and \mathbf{s}_y , respectively. Rotations about appropriate axes by an angle $\pi/2$ map them to the z basis:

$$|s, \mu_z\rangle = e^{-i\pi \mathbf{s}_y/2} |s, \mu_x\rangle = e^{i\pi \mathbf{s}_x/2} |s, \mu_y\rangle. \quad (2)$$

A measurement of the intensities $\{|\langle s, \mu_z | \psi \rangle|^2\}$ does not fix a single state $|\psi\rangle$ since the phases of the coefficients ψ_{μ_z} remain undetermined. However:
a spin state $|\psi\rangle \in \mathbf{H}^s$ is determined unambiguously if $3(2s + 1)$ probabilities

$$p(\mu_k) = |\psi_{\mu_k}|^2, \quad k = x, y, z, \quad (3)$$

are measured with a Stern-Gerlach apparatus along three axes not in a plane. For some exceptional states of measure zero in Hilbert space \mathbf{H}^s , the probabilities $p(\mu_k)$ might be compatible with a finite number of states.

For simplicity, the proof is carried out for orthogonal axes, the generalization being straightforward. Measuring with respect to *two* axes provides $2(2s+1)$ intensities which are usually compatible with a huge number of isolated states, in agreement with the result of [15]: the parameters fulfill nonlinear relations which may have multiple solutions. Enumerating the ensemble of possible “partner” states is complicated, so a distinctive third measurement is included from the very beginning.

It is useful to rephrase the statement at stake differently. According to (3) a state $|\psi\rangle$ gives rise to the *same* intensities as does $|\tilde{\psi}\rangle$ if its coefficients $\tilde{\psi}_{\mu_k} = \langle s, \mu_k | \tilde{\psi} \rangle$ differ from ψ_{μ_k} by phase factors only. Using (1) one writes thus

$$\sum_{\mu_k=-s}^s \psi_{\mu_k} e^{i\chi_k(\mu_k)} |s, \mu_k\rangle = \exp[i\chi_k(\mathbf{s}_k)] |\psi\rangle, \quad (4)$$

with three polynomials $\chi_k(\mu)$ of order $2s$ in μ at most. From now on, the index k is understood to take the values x, y , and z throughout. The coefficients in (4) thus define three states $|\psi_k\rangle = W_k^s |\psi\rangle$, where $W_k^s = \exp[i\chi_k(\mathbf{s}_k)]$ is a unitary operator diagonal in the k basis. Consequently, a state $|\tilde{\psi}\rangle$ compatible with (3) exists if and only if there are nontrivial unitary operators W_k^s such that

$$W_x^s |\psi\rangle = W_y^s |\psi\rangle = W_z^s |\psi\rangle \equiv |\tilde{\psi}\rangle. \quad (5)$$

It will turn out that this relation is satisfied only if the operators W_k^s are multiples of the identity, implying that $|\tilde{\psi}\rangle$ and $|\psi\rangle$ represent the *same* ray in Hilbert space.

Before turning to the proof, the intensities $p(\mu_k)$ in (3) are represented in a more compact way. Define three functions $m_k(\alpha)$ of a complex variable $\alpha \in \mathbf{C}$ by

$$m_k(\alpha) = \langle \psi | U_k^s(\alpha) | \psi \rangle \equiv \sum_{\mu_k=-s}^s e^{i\mu_k \alpha} p(\mu_k), \quad (6)$$

where the operator $U_k^s(\alpha) = \exp(i\alpha \mathbf{s}_k)$ rotates a state $|\psi\rangle$ about the k axis if $\alpha \in \mathbf{R}$. Eq. (6) is inverted easily using the orthogonality of the functions $\exp[-i\mu_k \alpha]$ on the interval $0 \leq \alpha < 2\pi$.

The proof showing that the data (3) are sufficient for state reconstruction is divided into five steps. (i) A 2^{2s} dimensional “parent” space \mathcal{H}^s is introduced which contains the Hilbert space \mathbf{H}^s of the spin s as a subspace. (ii) To each state $|\psi\rangle \in \mathbf{H}^s$ an equivalence class of product states $\{|\Psi\rangle \in \mathcal{H}^s\}$ is associated. (iii) A natural definition of *generic* states emerges for *product* states in \mathcal{H}^s and, *a fortiori*, in \mathbf{H}^s . (iv) An appropriate set of expectation values of the parent states $|\Psi\rangle$ fixes them uniquely. (v) Finally, it is shown that all states $|\tilde{\psi}\rangle$ satisfying (5) have parents in the *same* equivalence class as the original $|\psi\rangle$. Consequently, the (generic) state $|\psi\rangle$ is the only one giving rise to the intensities (3).

(i) The 2^{2s} dimensional “parent” space \mathcal{H}^s of \mathbf{H}^s is obtained from tensoring $2s$ copies of the Hilbert space \mathbf{C}^2 of a spin $1/2$:

$$\mathcal{H}^s = \bigotimes_{r=1}^{2s} \mathbf{C}_r^2. \quad (7)$$

A basis of \mathbf{C}^2 is given by the eigenstates $|\sigma\rangle \equiv |s = 1/2, \mu_3 = \sigma/2\rangle, \sigma = \pm 1$, of the third component of the spin $1/2$: $\sigma_3 |\sigma\rangle = \sigma |\sigma\rangle$. This choice induces a basis of \mathcal{H}^s formed by all product states

$$|\{\sigma_r\}\rangle = \bigotimes_{r=1}^{2s} |\sigma_r\rangle. \quad (8)$$

The parent space \mathcal{H}^s decomposes into a subspace $\mathcal{H}_{\text{sym}}^s$ and its complement,

$$\mathcal{H}^s = \mathcal{H}_{\text{sym}}^s \oplus (\mathcal{H}_{\text{sym}}^s)^\perp, \quad (9)$$

where $\mathcal{H}_{\text{sym}}^s$ is spanned by the $(2s+1)$ states obtained from completely symmetrizing those in (8):

$$\begin{aligned} |s, \mu_3\rangle &= \mathcal{S}_{2s} |\{\sigma_r\}\rangle \\ &\equiv N_{\mu_3}^s \sum_{\{\sigma_r\}} \delta(\sigma_1 + \dots + \sigma_{2s} - 2\mu_3) |\{\sigma_r\}\rangle, \end{aligned} \quad (10)$$

where $-s \leq \mu_3 \leq s$, using a symmetrizer of $2s$ objects, \mathcal{S}_{2s} , and the normalization factor $N_{\mu_3}^s = ((s - \mu_3)!(s + \mu_3)!/(2s)!)^{1/2}$. The space $\mathcal{H}_{\text{sym}}^s$ is important here because it carries a $(2s+1)$ dimensional irreducible representation of the group of rotations, $SU(2)$, obtained upon reducing the product representation [16]

$$\mathcal{U} |\{\sigma_r\}\rangle = \bigotimes_{r=1}^{2s} \sum_{\sigma'_r=\pm 1} |\sigma'_r\rangle \langle \sigma'_r | u_r | \sigma_r \rangle, \quad (11)$$

where u_r is the r -th copy of a rotation $u \in SU(2)$ of the fundamental representation acting on \mathbf{C}^2 , and \mathcal{U} is an operator defined on \mathcal{H}^s . Since Hilbert spaces of the same dimension are isomorphic, $\mathcal{H}_{\text{sym}}^s$ and \mathbf{H}^s will be identified from now on.

(ii) There is a one-to-one relation between states $|\psi\rangle \in \mathcal{H}_{\text{sym}}^s$ and equivalence classes of *product* states $|\Psi\rangle \in \mathcal{H}^s$:

$$|\Psi\rangle \equiv |\{\Psi^r\}\rangle = \bigotimes_{r=1}^{2s} \left(\sum_{\sigma_r} \Psi_{\sigma_r}^r |\sigma_r\rangle \right). \quad (12)$$

The equivalence relation \sim is defined as follows: the projection of a state $|\Psi\rangle$ in (12) onto a basis state $|s, \mu_3\rangle \in \mathcal{H}_{\text{sym}}^s$ must equal the corresponding expansion coefficient of $|\psi\rangle$ in the z basis, i.e.,

$$\langle s, \mu_3 | \Psi \rangle = N_\psi \langle s, \mu_z | \psi \rangle, \quad -s \leq \mu_3 = \mu_z \leq s, \quad (13)$$

and the factor $N_\psi > 0$ may depend on the state $|\psi\rangle$ under consideration but *not* on the index μ_z . Thus, $|\Psi\rangle \sim |\Psi'\rangle$

means that for a fixed $|\psi\rangle$, the Eqs. (13) hold for both product states, $|\Psi\rangle \sim |\Psi'\rangle$. The association of spin states $|\psi\rangle$ with product or “parent” states $|\Psi\rangle$ is essential for the following.

In order to determine the class of states satisfying Eq. (13) for a prescribed vector $|\psi\rangle$ (with definite phase), multiply by the factor $1/N_\mu^s$, by powers $(-z)^{\mu+s}$ and sum all terms. The right-hand-side then defines an analytic function

$$f_R(z) = N_\psi \sum_{\mu=-s}^s \frac{(-z)^{\mu+s}}{N_\mu^s} \psi_\mu \propto \prod_{r=1}^{2s} (z_r - z), \quad (14)$$

specified by the location of its $2s$ zeroes z_r in the complex plane. The left-hand-side yields a second analytic function of z ,

$$\begin{aligned} f_L(z) &= \sum_{\mu=-s}^s \sum_{\{\sigma_r\}} (-z)^{\mu+s} \delta(\sigma_1 + \dots + \sigma_{2s} - 2\mu) \Psi_{\sigma_1}^1 \dots \Psi_{\sigma_{2s}}^{2s} \\ &\equiv \prod_{r=1}^{2s} (\Psi_-^r - z \Psi_+^r), \quad \Psi_\pm^r \equiv \Psi_{\pm 1}^r. \end{aligned} \quad (15)$$

The $(2s+1)$ equations (13) are satisfied if $f_L(z)$ and $f_R(z)$ coincide. Being two polynomials of degree $2s$, this requires them to have identical zeroes,

$$\frac{\Psi_-^r}{\Psi_+^r} = z_r, \quad r = 1, \dots, 2s; \quad (16)$$

in addition, $f_L(0) = f_R(0)$ must hold. Due to the normalization $\langle \Psi^r | \Psi^r \rangle = |\Psi_+^r|^2 + |\Psi_-^r|^2 = 1$, one can write

$$\begin{pmatrix} \Psi_+^r \\ \Psi_-^r \end{pmatrix} = \frac{e^{i\kappa_r}}{\sqrt{1+|z_r|^2}} \begin{pmatrix} 1 \\ z_r \end{pmatrix}, \quad \kappa_r \in [0, 2\pi). \quad (17)$$

Thus, there are $2s$ undetermined phase factors $e^{i\kappa_r}$ with a product equal to 1 (remember that $|\psi\rangle$ denotes a *vector*). However, the overall ambiguity is even larger: when comparing the zeroes of the functions $f_L(z)$ and $f_R(z)$, there is no rule which would indicate what order to choose when writing down the product state $|\{\Psi^r\}\rangle$. In other words, the equivalence class of states defined by (13) consists of all states with coefficients (17) distributed in any order over the $2s$ spinors in (12). All these states are parents of the same $|\psi\rangle$ since they satisfy Eq. (13).

A given product state $|\Psi\rangle$ with components

$$\langle \{\sigma_r\} | \Psi \rangle = \Psi_{\{\sigma_r\}} = \prod_{r=1}^{2s} \Psi_{\sigma_r}^r, \quad (18)$$

has a unique “daughter” $|\psi\rangle$ to be read off directly. Upon parametrizing each factor $|\Psi^r\rangle$ by a complex number z_r ,

$$\begin{pmatrix} \Psi_+^r \\ \Psi_-^r \end{pmatrix} = \frac{1}{\sqrt{1+|z_r|^2}} \begin{pmatrix} 1 \\ z_r \end{pmatrix}, \quad (19)$$

one sees that the ensemble $\{z_r\} \equiv (z_1, \dots, z_{2s})$ (*no* order implied) defines the daughter $|\psi\rangle$ completely while a maximum of $(2s)!$ different parent states $|\Psi\rangle$ is associated with a given set $\{z_r\}$.

(iii) Suppose that three ensembles of $2s$ real numbers each, $\{x_r\}$, $\{y_r\}$, and $\{|z_r|\} \equiv (|z_1|, \dots, |z_{2s}|)$ with $z_r = x_r + iy_r$ are given in disorder. If one is able to construct the disordered ensemble of $2s$ complex numbers $\{z_r = x_r + iy_r\}$ upon using the $2s$ conditions $|z_r|^2 = x_r^2 + y_r^2$, the equivalence class with representative $|\Psi\rangle$ is called *generic*. In other words, it must be possible to combine unambiguously real and imaginary parts into complex numbers z_r . In this spirit, a daughter $|\psi\rangle \in \mathcal{H}_{\text{sym}}^s$ will be called *generic* if it has generic parents $|\Psi\rangle$. The procedure does not work if equalities such as $x_r = \pm y_{r'}, r \neq r'$ exist; hence *exceptional* states have measure zero.

(iv) It is shown now that the expectation values of rotations $\mathcal{U}_k(\alpha)$ about the axes x, y , and z , fix generic product states $|\Psi\rangle = |\{\Psi^r\}\rangle$ up to a permutation of the factors $|\Psi^r\rangle$ and an overall phase factor. A generic $|\Psi\rangle \in \mathcal{H}^s$ leads to three expectation values

$$M_k(\alpha) = \langle \Psi | \mathcal{U}_k(\alpha) | \Psi \rangle \equiv \prod_{r=1}^{2s} \langle \Psi^r | u_k^r(\alpha) | \Psi^r \rangle, \quad (20)$$

where $u_k(\alpha) = \mathbf{1} \cos(\alpha/2) + \boldsymbol{\sigma}_k \sin(\alpha/2)$ represents a rotation about axis k in \mathbf{C}^2 . Using the parametrization of Eq. (19), the functions $M_k(\alpha)$ defined in (20) read explicitly

$$M_x(\alpha) = \prod_{r=1}^{2s} \frac{\cos(\alpha/2) + 2ix_r \sin(\alpha/2)}{1 + |z_r|^2}, \quad (21a)$$

$$M_y(\alpha) = \prod_{r=1}^{2s} \frac{\cos(\alpha/2) + 2iy_r \sin(\alpha/2)}{1 + |z_r|^2}, \quad (21b)$$

$$M_z(\alpha) = \prod_{r=1}^{2s} \frac{\cos(\alpha/2) + i(1 - |z_r|^2) \sin(\alpha/2)}{1 + |z_r|^2}, \quad (21c)$$

where again $z_r = x_r + iy_r$. Denote by $|\tilde{\Psi}\rangle \equiv |\{\tilde{\Psi}^r\}\rangle$ another product state with expectations $\widetilde{M}_k(\alpha)$:

$$\widetilde{M}_k(\alpha) = \langle \tilde{\Psi} | \mathcal{U}_k(\alpha) | \tilde{\Psi} \rangle \equiv \prod_{r=1}^{2s} \langle \tilde{\Psi}^r | u_k^r(\alpha) | \tilde{\Psi}^r \rangle. \quad (22)$$

Upon describing the state $|\tilde{\Psi}\rangle$ by the sequence $\{\tilde{z}_r\}$, the three functions $\widetilde{M}_k(\alpha)$ are given by Eqs. 21) after replacing each z_r by \tilde{z}_r . It is shown now that the conditions

$$\langle \tilde{\Psi} | \mathcal{U}_k(\alpha) | \tilde{\Psi} \rangle = \langle \Psi | \mathcal{U}_k(\alpha) | \Psi \rangle, \quad k = x, y, z. \quad (23)$$

necessitate $|\tilde{\Psi}\rangle \sim |\Psi\rangle$. Being analytic in the complex α plane, the functions $M_k(\alpha)$ and $\tilde{M}_k(\alpha)$ are equal if they have same zeroes. The equality $\tilde{M}_z(\alpha) = M_z(\alpha)$ requires $|\tilde{z}_r| = |z_r|$. The condition $\tilde{M}_x(\alpha) = M_x(\alpha)$ in turn implies $\tilde{x}_r = x_r$; finally, $\tilde{y}_r = y_r$ follows from $\tilde{M}_y(\alpha) = M_y(\alpha)$. However, this procedure determines the ensembles $\{x_r\}$, $\{y_r\}$, and $\{|z_r|\}$ *without* any order of its members. Nevertheless, one can reconstruct the ensemble $\{z_r\}$ (*no* order implied) according to (iii) if $|\Psi\rangle$ is *generic* providing thus a *unique* equivalence class. For exceptional states, the $2s$ complex numbers cannot be reconstructed unambiguously since they might allow for parents contained in different equivalence classes.

(v) The results (i) to (iv) imply that the probabilities $p(\mu_k)$ for three directions x, y , and z as given in Eq. (3) determine a generic state $|\psi\rangle$ unambiguously. According to Eq. (5), a state $|\tilde{\psi}\rangle$ gives rise to the same probabilities as does $|\psi\rangle$ if one has

$$|\psi_x\rangle = |\psi_y\rangle = |\psi_z\rangle = |\tilde{\psi}\rangle. \quad (24)$$

For parent states $|\Psi_k\rangle$ of $|\psi_k\rangle$ this relation says that

$$|\Psi_x\rangle \sim |\Psi_y\rangle \sim |\Psi_z\rangle \sim |\tilde{\Psi}\rangle. \quad (25)$$

This implies that the mean values $\langle\Psi_k|\mathcal{U}_x(\alpha)|\Psi_k\rangle$ of the operator $\mathcal{U}_x(\alpha) = \otimes_r \exp[i\alpha\sigma_x/2]$ are equal for $k = x, y, z$: as products they are invariant under a permutation of their factors. This also holds for expectation values of the operators $\mathcal{U}_y(\alpha)$ and $\mathcal{U}_z(\alpha)$. Write the parent states $|\Psi_k\rangle$ in the form $\mathcal{W}_k|\Psi\rangle$ with operators $\mathcal{W}_k(\{\alpha_{k,r}\}) = \otimes_r \exp[i\alpha_{k,r}\sigma_k/2]$ defined on the parent space \mathcal{H}^s such that they have W_k^s as component acting in $\mathcal{H}_{\text{sym}}^s$. Contrary to the rotations $\mathcal{U}_k(\alpha)$ which depend linearly on the generators \mathbf{s}_k , the operators \mathcal{W}_k^s are *non-linear* functions $\chi(\mathbf{s}_k)$ of them, Eq. (4). Therefore, the operators $\mathcal{W}_k(\{\alpha_{k,r}\})$ depend on a set of $2s$ *different* angles $\{\alpha_{k,r}\}$. Using (25) one concludes

$$\begin{aligned} \langle\tilde{\Psi}|\mathcal{U}_k|\tilde{\Psi}\rangle &= \langle\Psi_k|\mathcal{U}_k|\Psi_k\rangle \\ &= \langle\Psi|\mathcal{W}_k^\dagger \mathcal{U}_k \mathcal{W}_k|\Psi\rangle = \langle\Psi|\mathcal{U}_k|\Psi\rangle. \end{aligned} \quad (26)$$

The third equality follows because \mathcal{W}_k and \mathcal{U}_k do commute, both being functions of \mathbf{s}_k only. Eq. (26) comes down to saying that the functions $M_k(\alpha)$ and $\tilde{M}_k(\alpha)$ coincide for all k and α . One concludes thus with (iv) that the state $|\tilde{\Psi}\rangle$, a parent of $|\tilde{\psi}\rangle$, is necessarily a member of the *same* equivalence class as the parent $|\Psi\rangle$ of $|\psi\rangle$. In other words, the application of the operators \mathcal{W}_k on a parent $|\Psi\rangle$ does not map it into another equivalence class. In the generic case, there is thus no state different from $|\psi\rangle$ with the same data (3) what was to be shown.

The reasoning (i) to (v) remains valid if one measures the intensities along directions characterized by unit vectors $\mathbf{n}_\zeta, \mathbf{n}_\eta$, and \mathbf{n}_ξ instead of three orthogonal axes.

These vectors must be linearly independent, that is, they have to span a *volume* in space: $\mathbf{n}_\zeta \cdot \mathbf{n}_\eta \times \mathbf{n}_\xi \neq 0$.

As a matter of fact, it is not excluded that the set of data (3) be also sufficient to determine exceptional states unambiguously. Suppose that the numbers $\{z_r\}$ are associated with a parent state $|\Psi\rangle$ and $\{z'_r\}$ with another one, $|\Psi'\rangle$, where both sets of complex numbers are obtained from the ensembles $\{x_r\}$ and $\{y_r\}$ through $|z_r|^2 = x_r^2 + y_r^2$. This does not necessarily imply the existence of an independent $|\psi'\rangle \neq |\psi\rangle$ since it is the basic conditions $|\psi'_{\mu_k}| = |\psi_{\mu_k}|$ which must be satisfied. Explicit calculations for low values of spin s show that this happens only if $\psi'_{\mu_k} = \psi_{\mu_k}^*$, resulting in $\langle\psi|(\mathbf{s}_y)^{2n+1}|\psi\rangle \equiv 0$ for all integers n . In any case one expects every non-genericity to vanish if the spatial directions involved are slightly modified.

To sum up, state reconstruction is possible if based on the $3(2s+1)$ moduli of the spin components with respect to three directions in space not all in the same plane. Compared to a constructive method using $(2s+1)^2$ real numbers, the non-constructive method presented here requires that considerably less parameters be determined experimentally, namely $3(2s+1)$.

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